THE POISSON INTEGRAL AND AN ANALYTIC FUNCTION ON ITS CIRCLE OF CONVERGENCE.

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Let $f(z)$ be an analytic function within the unit circle, having on the circumference C of this circle a finite number of singularities of logarithmic order, or of an order lower than that of a simple pole. Around the singular points on C we describe, in the interior of the circle of convergence, arcs of small circles of radii ϵ_p, and apply to these the process $\lim \epsilon_p \to 0$. We are then led to the conclusion that the integration of the Cauchy integral

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - \alpha}\, dz \tag{1}$$

may be carried out over the singularities.

Let us consider the analytic function $f(1/\bar{z})/\bar{z}(\bar{z} - \bar{\alpha})$ for values $|\bar{z}| > 1$, $\bar{\alpha}$ being a point within the unit circle.

From Cauchy's theorem we have

$$\frac{1}{2\pi i}\int_C \frac{f(1/\bar{z})}{\bar{z}(\bar{z} - \bar{\alpha})}\, d\bar{z} = 0; \tag{2}$$

or

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{\bar{z} - \bar{\alpha}}\, d\bar{z} - \frac{1}{2\pi i}\int_C \frac{f(z)}{\bar{z}}\, d\bar{z} = 0. \tag{2'}$$

From (1) and (2′) we get by subtraction

$$f(\alpha) = \frac{1}{2\pi}\int_C f(z)\left\{\frac{e^{i\theta}}{e^{i\theta} - Re^{i\psi}} + \frac{e^{-i\theta}}{e^{-i\theta} - Re^{-i\psi}}\right\} d\theta - \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\, dz.$$

Placing $f(\alpha) = U(R, \psi) + iV(R, \psi)$, we have

$$\begin{aligned} U(R, \psi) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{U(1, \theta)\{2 - 2R\cos(\psi - \theta)\}}{1 + R^2 - 2R\cos(\psi - \theta)}\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} U(1, \theta)\, d\theta. \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{U(1, \theta)(1 - R^2)}{1 + R^2 - 2R\cos(\psi - \theta)}\, d\theta, \end{aligned} \tag{3}$$

and a similar expression for $V(R, \psi)$. This is the well-known form of the integral of Poisson, except that now $U(1, \theta)$ may have logarithmic singularities, as well as algebraic singularities of an order lower than the first. When a singularity θ_1 is reached, we include this in an interval $\theta_1 - \epsilon$ to $\theta_1 + \epsilon$ and perform the operation $\lim \epsilon \to 0$. The integral over this in-

terval will then vanish. From formulas (1) and (2) we obtain the expression

$$\frac{1}{n!}f^{(n)}(0) = \frac{1}{2\pi}\int_0^{2\pi} f(z)\{e^{-in\theta} \pm e^{in\theta}\}d\theta,$$

where $f^{(n)}(0)$ denotes the nth derivative of $f(z)$ in the point $z = 0$.

Placing $f^{(n)}(0)/n! = \alpha_n + i\beta_n$, we have

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} U(1, \theta)\cos n\theta d\theta = \frac{1}{\pi}\int_0^{2\pi} V(1, \theta)\sin n\theta d\theta,$$

$$\beta_n = \frac{1}{\pi}\int_0^{2\pi} V(1, \theta)\cos n\theta d\theta = -\frac{1}{\pi}\int_0^{2\pi} U(1, \theta)\sin n\theta d\theta,$$

which are the well-known values of the coefficients of the Fourier series.

If a point α be now moved into a regular point of $f(z)$ on the circumference of the circle of convergence, we shall have for this point

$$\text{(1')} \qquad \frac{1}{2}f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - \alpha}dz,$$

and

$$\text{(2'')} \qquad \frac{1}{2}f\left(\frac{1}{\bar{\alpha}}\right) = \frac{1}{2\pi i}\int_C \frac{f(z)}{\bar{z} - \bar{\alpha}}d\bar{z} - \frac{1}{2\pi i}\int_C \frac{f(z)}{\bar{z}}d\bar{z},$$

from which follows

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} f(z)\left\{\frac{e^{i\theta}}{e^{i\theta} - e^{i\psi}} + \frac{e^{-i\theta}}{e^{-i\theta} - e^{-i\psi}}\right\}d\theta \\ &\qquad\qquad - \frac{1}{2\pi}\int_0^{2\pi}\{U(1, \theta) + iV(1, \theta)\}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z)\left[1 + \sum_1^n e^{im(\psi-\theta)} + 1 + \sum_1^n e^{-im(\psi-\theta)}\right]d\theta + \\ &\qquad + \frac{1}{2\pi}\int_0^{2\pi} f(z)\left\{\frac{e^{i(n+1)(\psi-\theta)}}{1 - e^{i(\psi-\theta)}} + \frac{e^{-i(n+1)(\psi-\theta)}}{1 - e^{-i(\psi-\theta)}}\right\}d\theta \\ &\qquad\qquad - \frac{1}{2\pi}\int_0^{2\pi}\{U(1, \theta) + iV(1, \theta)\}d\theta, \end{aligned}$$

or

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} f(z)d\theta + \sum_1^n \frac{1}{\pi}\int_0^{2\pi} f(z)\cos m(\psi - \theta)d\theta \\ &\qquad + \frac{i}{2\pi}\int_0^{2\pi} f(z)\left\{\frac{e^{i(n+1)(\psi-\theta)}}{2\sin\frac{\psi - \theta}{2}e^{i\frac{\psi-\theta}{2}}} - \frac{e^{-i(n+1)(\psi-\theta)}}{2\sin\frac{\psi - \theta}{2}e^{-i\frac{\psi-\theta}{2}}}\right\}d\theta. \end{aligned}$$

That is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{U(1,\,\theta)\sin(2n+1)\dfrac{\psi-\theta}{2}}{\sin\dfrac{\psi-\theta}{2}}d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}U(1,\,\theta)d\theta+\sum_1^n\frac{1}{n}\int_0^{2\pi}U(1,\,\theta)\cos m(\psi-\theta)d\theta. \quad (4)$$

This is the familiar summation formula for the common Fourier series. A similar expression is obtained for $V(1,\,\theta)$.

Since for a value θ_1 of θ $U(1,\,\theta)$ can have only a singularity of lower order than the extension of the interval of integration ϵ, or only a singularity of an order lower than the linear, we may apply the general theory of Fourier series to the integral on the left hand side of equation (4). We have then for every regular place ψ of $U(1,\,\theta)$

$$U(1,\,\psi)=\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{U(1,\,\theta)\sin(2n+1)\dfrac{\psi-\theta}{2}}{\sin\dfrac{\psi-\theta}{2}}d\theta. \quad (5)$$

An addition of (1′) and (2″) would, on account of (4) and (5), have led to the formula

$$0=\lim_{n\to\infty}\int_0^{2\pi}\frac{U(1,\,\theta)\cos(2n+1)\dfrac{\psi-\theta}{2}}{\sin\dfrac{\psi-\theta}{2}}d\theta \quad (6)$$

which could also have been proved directly.

From these results we conclude conversely that an analytic function which has only a finite number of singularities on its circle of convergence, these singularities being of logarithmic order or of order lower than that of a simple pole, may be represented in every regular point by the familiar series which is derived by means of Cauchy's integral and which is valid within the circle of convergence. This fact, it seems, is equivalent to the contents of a theorem by Fatou-M. Riesz.*

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* E. Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. Berlin, 1916.